Central Force

Central force is a force which acts on a particle s.t. it is always directed towards or away from a fixed centre and the magnitude of the force depends only in the distance from the centre.

It is represented by $\vec{F} = f(r)\hat{r}$

where, $f(r)$ = function of distance r

$\hat{r}$ = unit vector along $\vec{r}$ from the fixed centre.

Motion under Central force :- "The equation of motion" :-

Let's r be the

Let's consider two bodies separated by a distance r. Since there's no external force acting on the system, the angular momentum of the system remain constant. Hence the rotation of lighter body around the heavier one will remain confined to a plane. Let r, θ be the co-ordinates of the body in plane polar-coordinate system at any instant. Its position vector at that instant,

$$\vec{r} = r\hat{e_r}$$

Velocity v of the body will be,

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e_r})$$

$$\Rightarrow v = \frac{dr}{dt}\hat{e_r} + r\frac{d\hat{e_r}}{dt} = \dot{r}\hat{e_r} + r\dot{\hat{e_r}}$$

It can be proved that,

$$\dot{\hat{e_r}} = \dot{\theta}\hat{e_\theta}$$

$$\therefore \quad \vec{v} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta} \qquad \text{——— (1)}$$

where, $\hat{e_r}$ & $\hat{r_\theta}$ are the unit vectors directed along increasing r and increasing θ.

The acceleration $\vec{a}$ of the body is,

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \frac{d}{dt}(\dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}) = (\ddot{r} - r\dot{\theta}^2)\hat{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e_\theta}$$

$$= \frac{d\dot{r}}{dt}\hat{e_r} + \dot{r}\dot{\hat{e_r}} + \dot{\hat{e_\theta}} \qquad \text{——— (2)}$$

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The eqn of motion of the body can be written as,

$$\mu \vec{a} = F(r)\hat{e}_r$$

where, μ = reduced mass of the system.

and $F(r) = -\frac{GMm}{r^2}$

Substituting for $\vec{a}$ from eqn (2) we have,

$$\mu(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + \mu(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta = F(r)\hat{e}_r$$

Comparing the co-efficients of orthogonal unit vector $\hat{e}_r$ & $\hat{e}_\theta$ on both sides,

$$\mu(\ddot{r} - r\dot{\theta}^2) = F(r) \quad \text{——— (3)}$$

$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \quad \text{——— (4)}$$

Eqn (4) indicates that there is no force acting on the body tangentially to increase θ. Only force acting on it is radially inwards. Therefore, eqn (3) gives the eqn of motion of the body. If L is the angular momentum of the body,

$$L = \mu r^2 \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{L}{\mu r^2}$$

Substituting for $F(r)$ and $\dot{\theta}$ in eqn (3),

$$\mu\left[\ddot{r} - r\left(\frac{L}{\mu r^2}\right)^2\right] = -\frac{\partial V}{\partial r}$$

$$\Rightarrow \mu\ddot{r} = \frac{L^2}{\mu r^3} - \frac{\partial V}{\partial r} \quad \text{——— (5)}$$

But $\frac{L^2}{\mu r^3} = -\frac{\partial}{\partial r}\left(\frac{L^2}{2\mu r^2}\right)$

Making this substitution in eqn (5) we have,

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{L^2}{2\mu r^2}\right) - \frac{\partial V}{\partial r}$$

$$\Rightarrow \mu\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{L^2}{2\mu r^2} + V\right)$$

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L, V and μ being constants, there is only one variable in the equation. Therefore $\frac{\partial}{\partial r}$ can be replaced by d/dr.

$$\therefore \mu\ddot{r} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} + V\right)$$

$$\Rightarrow \mu\ddot{r}\frac{dr}{dt} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} + V\right)\frac{dr}{dt} \quad \left[\text{Multiplying } \frac{dr}{dt} \text{ both sides}\right]$$

$$\Rightarrow \mu\ddot{r}\dot{r} = -\frac{d}{dt}\left(\frac{L^2}{2\mu r^2} + V\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}\left(\frac{L^2}{2\mu r^2} + V\right)$$

Integrating both sides,

$$\frac{1}{2}\mu\dot{r}^2 = -\left(\frac{L^2}{2\mu r^2} + V\right) + E$$

where, E is the constant of integration.

$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V \quad \text{——— (6)}$$

Here $\frac{1}{2}\mu\dot{r}^2$ represents kinetic energy, $L^2/2\mu r^2$ centrifugal potential energy and V is the potential energy. Thus eq^n (6) is the statement of conservation of energy with E as the total energy of the system.

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

where, $U_{eff}(r)$ = effective potential energy

$$= \frac{L^2}{2\mu r^2} + V$$

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